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The quantum one loop trace anomaly of the higher spin conformal conserved currents in the bulk of AdS_4

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ABSTRACT

An analysis of the structure and singularities of the one loop two point function of the higher spin traceless and conserved currents constructed from the single scalar field in AdS space is presented. The detailed renormalization procedure is constructed and the quantum violation of the traceless Ward identity is investigated. The connection with the one loop effective action for higher spin gauge fields is discussed.

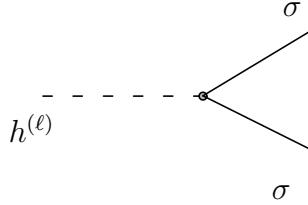
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1 Introduction

The increasing interest in the complicated problem of quantization and interaction of the higher spin gauge theories in AdS space [1, 2] is connected with the AdS_4/CFT_3 correspondence of the critical $O(N)$ sigma model and four dimensional higher spin gauge theory in anti de Sitter space ($HS(4)$) proposed in [3]. This special case of general AdS/CFT correspondence can be used for direct reconstruction of the unknown bulk interaction from the well developed boundary theory [4, 5]. This unique case is interesting also in view of the properties of the renormalization group flow from the free field *unstable* point of the boundary $O(N)$ vector model with the stable critical interacting conformal point in the large $-N$ limit by the deformation with the double trace marginal operator. This flow should correspond to the quantum behaviour on the bulk side of the same higher spin theory ($HS(4)$) and different boundary conditions for the quantized scalar field [6]. Note that in the second and nontrivial conformal point of the $d = 3$ sigma model all higher spin currents except the energy-momentum tensor (spin two) are conserved only in the large N limit and their divergence is of first order in $\frac{1}{N}$. On the bulk side this must correspond to a certain mass generation mechanism on the one loop level (again order of $\frac{1}{N}$) of the interacting $HS(4)$ gauge theory. This mass generation mechanism was considered in our previous articles [7, 8]. In this article we want to consider the bulk mirror of the instability of the free field conformal point of the boundary theory. The main idea is the following: Because any interaction pulls the free theory out of the conformal point with an infinite number of the higher even spin conserved traceless currents (corresponding to the traceless higher spin gauge fields on the bulk) any one loop self energy graph constructed using any possible (self)interaction of the corresponding gauge higher spin field on the bulk should violate tracelessness of the latter. Here we will investigate only the simplest possible minimal gauge field times current interaction of the HS field with the bulk scalar field constructed in [9] and consider the behaviour of the short distance singularities in the coordinate space of the corresponding one loop two point function of the conserved currents responsible for the renormalization of the propagator of the HS gauge fields. We prove that correct regularization and renormalization leads to the violation of the traceless Ward identities when we maintain the quantum level conservation Ward identities (quantum gauge invariance). We call this phenomenon *Higher spin trace anomaly* due to the analogy with the conformal trace anomaly of the energy momentum tensor which is the spin equal two case of our general spin consideration.

2 Conserved current in AdS

Using our notation of the previous papers [8, 9, 10] we consider the minimal interaction of the conformal higher spin field with the conserved traceless current constructed from the conformally coupled scalar in AdS_4 space[†].



$$= \quad S_{int}^{(\ell)conf} = \frac{1}{\ell} \int d^4x \sqrt{g} h^{(\ell)\mu_1 \dots \mu_\ell} J_{\mu_1 \dots \mu_\ell}^{(\ell)}.$$

Here $h^{(\ell)}$ is the spin ℓ gauge field and $J^{(\ell)}(a; z) = J_{\mu_1 \dots \mu_\ell}^{(\ell)} a^{\mu_1} \dots a^{\mu_\ell}$ the conserved traceless current constructed from the conformally coupled scalar $\sigma(z)$ [9] in $D = d + 1$ dimensional AdS space

$$\begin{aligned} J^{(\ell)}(z; a) &= \frac{1}{2} \sum_{p=0}^{\ell} A_p (a\nabla)^{\ell-p} \sigma(z) (a\nabla)^p \sigma(z) \\ &+ \frac{a^2}{2} \sum_{p=1}^{\ell-1} B_p (a\nabla)^{\ell-p-1} \nabla_\mu \sigma(z) (a\nabla)^{p-1} \nabla^\mu \sigma(z) \\ &+ \frac{a^2}{2L^2} \sum_{p=1}^{\ell-1} C_p (a\nabla)^{\ell-p-1} \sigma(z) (a\nabla)^{p-1} \sigma(z) + O(a^4) + O\left(\frac{1}{L^4}\right), \end{aligned} \quad (1)$$

where $A_p = A_{\ell-p}$, $B_p = B_{\ell-p}$, $C_p = C_{\ell-p}$ and $A_0 = 1$. The tracelessness condition

$$\square_a J^{(\ell)}(z; a) = \frac{\partial^2}{\partial a_\mu \partial a^\mu} J^{(\ell)}(z; a) = 0, \quad (2)$$

[†]We will use Euclidian AdS_{d+1} with conformal flat metric, curvature and covariant derivatives satisfying

$$\begin{aligned} ds^2 &= g_{\mu\nu}(z) dz^\mu dz^\nu = \frac{L^2}{(z^0)^2} \delta_{\mu\nu} dz^\mu dz^\nu, \quad \sqrt{g} = \frac{L^{d+1}}{(z^0)^{d+1}}, \\ [\nabla_\mu, \nabla_\nu] V_\lambda^\rho &= R_{\mu\nu\lambda}{}^\sigma V_\sigma^\rho - R_{\mu\nu\sigma}{}^\rho V_\lambda^\sigma, \\ R_{\mu\nu\lambda}{}^\rho &= -\frac{1}{(z^0)^2} (\delta_{\mu\lambda} \delta_\nu^\rho - \delta_{\nu\lambda} \delta_\mu^\rho) = -\frac{1}{L^2} (g_{\mu\lambda}(z) \delta_\nu^\rho - g_{\nu\lambda}(z) \delta_\mu^\rho), \\ R_{\mu\nu} &= -\frac{d}{(z^0)^2} \delta_{\mu\nu} = -\frac{d}{L^2} g_{\mu\nu}(z), \quad R = -\frac{d(d+1)}{L^2}. \end{aligned}$$

For shortening the notation and calculation we contract all rank ℓ symmetric tensors with the ℓ -fold tensor product of a vector a^μ .

fixes relations between B_p , C_p and A_p in the following way [11]

$$B_p = -\frac{p(\ell-p)}{(D+2\ell-4)} A_p, \quad (3)$$

$$C_p = \frac{-1}{2(D+2\ell-4)} [s_t(p+1, \ell, D)A_{p+1} + s_t(\ell-p+1, \ell, D)A_{p-1}], \quad (4)$$

$$s_t(p, \ell, D) = \frac{1}{4}p(p-1)D(D-2) + \frac{1}{3}p(p-1)(p-2)(\ell+2D-5). \quad (5)$$

The unknown A_p can be fixed using the conservation condition for the current

$$\nabla \cdot \partial_a J^{(\ell)}(z; a) = \nabla^\mu \frac{\partial}{\partial a^\mu} J^{(\ell)}(z; a) = 0. \quad (6)$$

This leads to a recursion relation with the same solution for the A_p coefficients [12] as in the flat $D = d + 1$ dimensional case

$$A_p = (-1)^p \frac{\binom{\ell}{p} \binom{\ell+D-4}{p+\frac{D}{2}-2}}{\binom{\ell+D-4}{\frac{D}{2}-2}}. \quad (7)$$

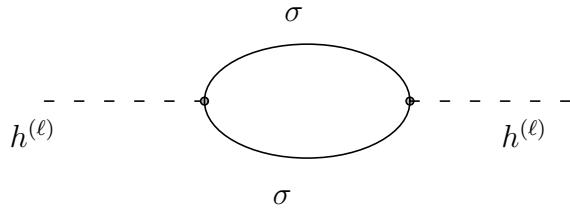
For the important case $D = 4$ this formula simplifies to

$$A_p = (-1)^p \binom{\ell}{p}^2. \quad (8)$$

The result of our previous consideration [11] was the following: the curvature corrections do not change flat space tracelessness and conservation conditions between leading coefficients and therefore the solution (7) remains valid. So we can concentrate in the future only on the first part of our currents described by the first set of coefficients (8)(A terms) in four dimensional *AdS* space

$$J^{(\ell)}(z; a) = \frac{1}{2} \sum_{p=0}^{\ell} A_p (a\nabla)^{\ell-p} \sigma(z) (a\nabla)^p \sigma(z) + \text{B\&C terms} \quad (9)$$

knowing for sure that all trace (B -) and curvature (C -) terms are not essential for quantum dynamics considered here and could be restored from kinematical considerations and commutation relations of covariant derivatives in *AdS* space. In the center of our interest here we will put the following self-energy one loop diagram constructed from the A term of our interaction current in coordinate space



This diagram is connected to the two point function of the currents in the standard way

$$\int_{z_1} \int_{z_2} h^{(\ell)}(z_1; a) \overleftarrow{\partial}_a \cdot \overrightarrow{\partial}_a < J^{(\ell)}(z_1; a) J^{(\ell)}(z_2; c) > \overleftarrow{\partial}_c \cdot \overrightarrow{\partial}_c h^{(\ell)}(z_2; c). \quad (10)$$

The quantum one loop behaviour, singularity and renormalization of this two point function will be explored in the next sections.

3 Loop function and Ward identity

For the calculation of the one loop two point function

$$\Pi^{(\ell)}(z_1; a|z_2; c) := < J^{(\ell)}(z_1; a) J^{(\ell)}(z_2; c) > \quad (11)$$

we have to insert corresponding A terms of currents (1) in (11) and apply just Wick's theorem. The propagator of the scalar field in AdS_4 quantized with a boundary condition corresponding to the free conformal point of the boundary $O(N)$ model is[‡]

$$< \sigma(z_1) \sigma(z_2) > = \frac{1}{8\pi^2} \left(\frac{1}{\zeta - 1} + \frac{1}{\zeta + 1} \right), \quad (12)$$

where

$$\zeta = \frac{(z_1^0)^2 + (z_2^0)^2 + (\vec{z}_1 - \vec{z}_2)^2}{2z_1^0 z_2^0} \quad (13)$$

and $\zeta - 1$ is the invariant geodesic distance in AdS . Due to this property our correlation function depends only on the AdS invariant geodesic distance and it's derivatives exactly as in the case of the higher spin propagator described in our previous article [10]. The general rule for working with such objects is analyzed in detail in the same article. The main point is the following. The tensorial structure of any two point function in AdS space can be described using a general basis of the independent bitensors [13], [14], [15], [16]

$$I_1(a, c) := (a\partial)_1(c\partial)_2 \zeta(z_1, z_2), \quad (14)$$

$$I_2(a, c) := (a\partial)_1 \zeta(z_1, z_2) (c\partial)_2 \zeta(z_1, z_2), \quad (15)$$

$$I_3(a, c) := a_1^2 I_{2c}^2 + c_2^2 I_{1a}^2, \quad (16)$$

$$I_4 := a_1^2 c_2^2, \quad (17)$$

$$I_{1a} := (a\partial)_1 \zeta(z_1, z_2), \quad I_{2c} := (c\partial)_2 \zeta(z_1, z_2), \quad (18)$$

$$(a\partial)_1 = a^\mu \frac{\partial}{\partial z_1^\mu}, \quad (c\partial)_2 = c^\mu \frac{\partial}{\partial z_2^\mu}, \quad (19)$$

$$a_1^2 = g_{\mu\nu}(z_1) a^\mu a^\nu, \quad c_2^2 = g_{\mu\nu}(z_2) c^\mu c^\nu. \quad (20)$$

[‡]From now on we put $L = 1$.

In this case this basis should appear automatically after contractions of scalars and action of the vertex derivatives. In general we have to get an expansion with all four basis elements

$$\Pi^\ell(z_1; a|z_2; c) = \Psi^\ell[F] + \sum_{n,m; 0 < 2(n+m) < \ell} I_3^n I_4^m \Psi^{\ell-2(n+m)}[G^{(n,m)}]. \quad (21)$$

Here we introduce a special map from the set $\{F_k(\zeta)\}_{k=0}^\ell$ of the $\ell + 1$ functions of ζ to the space of $\ell \times \ell$ bitensors

$$\Psi^\ell[F] = \sum_{k=0}^\ell I_1^{\ell-k}(a, c) I_2^k(a, c) F_k(\zeta). \quad (22)$$

But because for the analysis of short distance singularities ($\zeta \rightarrow 1$) only the A terms of currents are important for us, we will restrict our consideration on the first part of (21) connected with the I_1, I_2 bitensors calling all monomials corresponding to I_3 and I_4 in the above sum and the corresponding sets of functions $\{G_k^{(n,m)}\}_{k=0}^{\ell-2(n+m)}$ the "trace terms"

$$\Pi^\ell(z_1; a|z_2; c) = \Psi^\ell[F] + \text{trace terms}. \quad (23)$$

So all our calculations will be with exception of $O(a^2)$ and $O(c^2)$ terms. These trace terms in principle can be analyzed using the computer program [17]. All important relations for the calculations to be performed below can be found in [10] and listed in Appendix A of this article. In the main text we will only present the so called general multigradient map for the scalar function of ζ

$$(c \cdot \nabla)_2^q (a \cdot \nabla)_1^p F(\zeta) = \sum_{n=0}^q \frac{p! \binom{q}{n}}{(p-q+n)!} F^{(p+q)}(\zeta) I_1^{q-n} I_{1a}^{p-q+n} I_{2c}^n + \text{traces}, \quad (24)$$

$$F^{(k)}(\zeta) := \frac{\partial^k}{\partial \zeta^k} F(\zeta) \quad , \quad p \geq q. \quad (25)$$

Now we are ready to calculate the correlation function (11). Substituting (9) in (11) and using (12), (8) and (24) after some manipulations we obtain the

following formula for our two point function

$$\Pi^\ell(z_1; a|z_2; c) = \frac{1}{2^7\pi^4} \sum_{k=0}^{\ell} I_1^{\ell-k} I_2^k R_k^\ell(\zeta) + \text{traces}, \quad (26)$$

$$R_k^\ell(\zeta) = \sum_{r=0}^{\ell+k} Q_{k,r}^\ell \left\{ \Phi_k^{sing}(\zeta) + \Phi_{r,k}^{mixed}(\zeta) + \Phi_k^{reg}(\zeta) \right\}, \quad (27)$$

$$Q_{k,r}^\ell = \frac{(-1)^{\ell+k} (\ell!)^2 (\ell+k)!}{(k!)^2 (\ell-k)!} \sum_{p,q=r-k}^r \frac{(-1)^{p+q} \binom{\ell}{p} \binom{\ell}{q} \binom{k}{r-p} \binom{k}{r-q} \binom{\ell-k}{p+q-r}}{\binom{l+k}{r}}, \quad (28)$$

$$\Phi_k^{sing}(\zeta) = \frac{1}{(\zeta-1)^{\ell+k+2}}, \quad (29)$$

$$\Phi_k^{reg}(\zeta) = \frac{1}{(\zeta+1)^{\ell+k+2}}, \quad (30)$$

$$\Phi_{k,r}^{mixed}(\zeta) = \frac{2}{(\zeta-1)^{r+1} (\zeta+1)^{\ell+k-r+1}} ; \quad (Q_{k,r}^\ell = Q_{k,\ell+k-r}^\ell), \quad (31)$$

where in (27) we separated the singular, regular and mixed parts in view of their short distance behaviour at $\zeta \rightarrow 1$.

For the investigation of the short distance singularities at $\zeta \rightarrow 1$ we have to take into account the following:

- $I_1(a, c; \zeta) \rightarrow \frac{(a \cdot c)}{(z_1^0)^2}$ if $\zeta \rightarrow 1$.
- $I_2(a, c; \zeta) \rightarrow 0$ but $\frac{I_2(a, c; \zeta)}{\zeta-1}$ is finite when $\zeta \rightarrow 1$.
- Singularities start from $(\zeta-1)^{-2}$ because in $D = 4$ $\sqrt{g}d^4z$ behaves as $(\zeta^2 - 1)d\zeta$ (will be shown in Appendix A).

The next important point is the singular part of the mixed terms coming from (31). These terms are expanded as

$$\Psi_{sing}^{(\ell)mixed} := \Psi^{(\ell)}[F_{sing}^{(\ell)}], \quad (32)$$

$$F_{k,sing}^{(\ell)} = \frac{1}{2^7\pi^4} \sum_{m=2}^{\ell+1} a_{k,m}^{(\ell)} (\zeta-1)^{-m-k}, \quad (33)$$

$$a_{k,m}^{(\ell)} = 2^{-\ell-1+m} \sum_{r=0}^{\ell+1-m} (-1)^{\ell+1-m-r} \binom{\ell+1-m}{r} Q_{k,r}^\ell. \quad (34)$$

In Appendix B we prove that the bitensor (32) formed by the singular mixed part can be expressed as a bigradient of a spin $\ell-1$ bitensor

$$\Psi_{sing}^{(\ell)mixed}[F_{sing}^{(\ell)}] = (a \cdot \nabla_1)(c \cdot \nabla_2) \Psi^{(\ell-1)}[G], \quad (35)$$

formed by the set of functions $G_k^{(\ell-1)}(\zeta)$. Moreover this procedure can be continued recursively in ℓ if we separate the singular part of $\Psi^{(\ell-1)}[G]$ and express the latter as a gradient term. The final formula is

$$\Psi_{sing}^{(\ell)mixed} = \sum_{n=1}^{\ell} [(a \cdot \nabla_1)(c \cdot \nabla_2)]^n \sum_{k=0}^{\ell-n} I_1^{\ell-n-k} I_2^k b_{k,n}^{(\ell-n)} (\zeta - 1)^{-k-1}. \quad (36)$$

Both (35) and (36) are derived in Appendix B. By (36) the singular part of $\Psi^{(\ell)mixed}$ is expressed as a sum of powers of bigradients applied to regular (integrable) functions. The whole expression (36) is therefore a well-defined distribution and does not need any regularization if we apply extracted derivatives on nonsingular external higher spin gauge fields in the effective action. This allows us to concentrate on the main singular part of the correlation function (26)

$$\Pi^\ell(a, c; \zeta) = \frac{1}{2^7 \pi^4} \sum_{k=0}^{\ell} \frac{I_1^{\ell-k} I_2^k}{(\zeta - 1)^{\ell+k+2}} \sum_{r=0}^{\ell+k} Q_{k,r}^\ell, \quad (37)$$

which can not be presented in such a form (35), (36).

The crucial point here is the possibility to sum the coefficients $Q_{k,r}^\ell$ because the main singularity (29) does not depend on the index r . Indeed we can observe the following important identity

$$\sum_{r=0}^{\ell+k} Q_{k,r}^\ell = (-1)^{\ell+k} (2\ell)! \binom{\ell}{k}, \quad (38)$$

or explicitly

$$\frac{(\ell+k)!}{(k!)^2 (\ell-k)!} \sum_{r=0}^{\ell+k} \sum_{p,q=r-k}^r \frac{(-1)^{p+q} \binom{\ell}{p} \binom{\ell}{q} \binom{k}{r-p} \binom{k}{r-q} \binom{\ell-k}{p+q-r}}{\binom{l+k}{r}} = \binom{2\ell}{\ell} \binom{\ell}{k}. \quad (39)$$

Unfortunately we have no analytic proof of identity (39) but we are absolutely sure that it is right because we have checked this strange identity for many possible numbers ℓ and k with a computer program (Mathematica 5). Thus using (38) we can immediately sum (37) and obtain the following nice relation

$$\Pi^\ell(a, c; \zeta) = \frac{(-1)^\ell (2\ell)!}{2^7 \pi^4} \left(I_1 - \frac{I_2}{\zeta - 1} \right)^\ell \frac{1}{(\zeta - 1)^{\ell+2}}. \quad (40)$$

The beauty of the expression (40) is the following: This main singular part of our loop function is satisfying the naive Ward identities following from the conservation and tracelessness conditions of our currents (2),(6) directly without contribution of the corresponding partner trace terms described by the expansion in the other two bitensors I_3, I_4

$$\square_a \Pi^\ell(a, c; \zeta) = (\nabla \cdot \partial_a) \Pi^\ell(a, c; \zeta) = 0. \quad (41)$$

Here Π^ℓ is considered as analytic function on ζ . In the next section we will introduce a correct regularization and renormalization of (40) and investigate the quantum violation of the tracelessness condition.

4 Extraction of singularities and renormalization

Now we introduce the correct regularization for extracting the singularities from (40). For convenience we can use instead of ζ as equivalent invariant variable - the chordal distance

$$u = \zeta - 1 = \frac{(z_1 - z_2)^2}{2z_1^0 z_2^0}. \quad (42)$$

For this variable the singularity is located at $u \rightarrow 0$. Then expanding again (40) in the form

$$\Pi^\ell(a, c; u) = \frac{(-1)^\ell (2\ell)!}{2^3 (4\pi)^4} \sum_{k=0}^{\ell} I_1^{\ell-k} I_2^k F_k^B(u), \quad (43)$$

$$F_k^B(u) = (-1)^k \binom{\ell}{k} \frac{1}{u^{\ell+k+2}}, \quad (44)$$

we realize that the main task is the extraction of the singularities from the *bare* distributions $F_k^B \sim u^{-n}, n \in \mathbb{N}$. This can be done by shifting the integer n by some infinitesimal amount ϵ i.e. $u^{\epsilon-n}$. Considering some smooth test function $f(u), u \in \mathbb{R}^+$ and using a Laplace transformation ($f(u) = \int_0^\infty ds e^{-us} \hat{f}(s)$) we can write

$$\int_0^\infty \frac{f(u)}{u^{n-\epsilon}} du = \int_0^\infty ds \hat{f}(s) s^{n-\epsilon-1} \Gamma(\epsilon - n + 1) \quad (45)$$

$$= \frac{\partial^{n-1}}{\partial u^{n-1}} f(u)|_{u=0} \left(\frac{1}{\epsilon(n-1)!} + \text{reg. part} \right). \quad (46)$$

So we see that the singular part of our distribution is the $n-1$ order derivative of the delta function

$$\left[\frac{1}{u^{n-\epsilon}} \right]_{sing} = \frac{1}{\epsilon} \frac{(-1)^{n-1}}{(n-1)!} \delta^{(n-1)}(u). \quad (47)$$

The next step is to connect this ϵ shifting with some *gauge invariant* scheme such as dimensional regularization for our bare correlator formed by the set of distributions (44) and the Ward identities (41). Finally we will preserve the Ward identity of current conservation necessary for gauge invariance and extract the violation of tracelessness for this case.

For doing this note that the set F_k^B satisfies the conservation and tracelessness conditions for $d = 3$ ($D = 4$) if we understand them as analytic functions of u . Using (A.28) and (A.30) for $d = 3$ we can see that

$$(Div_\ell F^B)_k^{d=3} = 0, \quad (Tr_\ell F^B)_k^{d=3} = 0. \quad (48)$$

On the other hand it is easy to see that we can satisfy Ward identities for general d if we regularize the bare distributions in the following way

$$F_k^R = (-1)^k \binom{\ell}{k} \frac{1}{u^{\ell+k+d-1}}, \quad (49)$$

which depends analytically on d with a pole at $d = 3$. Actually we need to check only the conservation condition (A.28) because tracelessness (A.30) does not include derivatives and dimension. Indeed

$$(Div_\ell F^R)_k = (\ell - k)(u + 1)\partial_u F_k^R + (k + 1)u(u + 2)\partial_u F_{k+1}^R + (\ell - k)(\ell + d + k)F_k^R + (k + 1)(\ell + d + k + 1)(u + 1)F_{k+1}^R = 0, \quad (50)$$

$$(Tr_\ell F^R)_k = (\ell - k)(\ell - k - 1)F_k^R + 2(k + 1)(\ell - k - 1)(u + 1)F_{k+1}^R + (k + 2)(k + 1)u(u + 2)F_{k+2}^R = 0, \quad (51)$$

hold analytically in d . So we can just put in (50),(51) $d = 3 - \epsilon$ and say that we constructed the regularized Ward identities.

Then the procedure is more or less standard. We can split (49) for $d = 3 - \epsilon$ in a singular and renormalized parts using (47)

$$F_k^R(u) = \binom{\ell}{k} \left(\frac{1}{\epsilon} \frac{(-1)^{\ell+1}}{(\ell + k + 1)!} \delta^{(\ell+k+1)}(u) + f_k(u) \right) + F_k^{Ren}(u), \quad (52)$$

where we included also a set of finite distributions $f_k(u)$ (without ϵ pole) to describe the finite renormalization freedom. Analyzing (52) we can say that our singular part corresponds to the local counterterms of the effective action because each is proportional to a derivative of the delta function. On the other side the renormalized correlation function formed by $F_k^{Ren}(u)$ will on the quantum level get the same trace as a subtracted singular part but with opposite sign because the regularized expression is traceless and conserved. So we can insert the subtraction parts

$$F_k^S(u) = \binom{\ell}{k} \left(\frac{1}{\epsilon} \frac{(-1)^{\ell+1}}{(\ell + k + 1)!} \delta^{(\ell+k+1)}(u) + f_k(u) \right) \quad (53)$$

in the regularized current conservation Ward identity (50) for $d = 3 - \epsilon$ and obtain equations for the set of $f_k(u)$ after sending $\epsilon \rightarrow 0$. Using the relation $u\delta^{(n)}(u) = -n\delta^{(n-1)}(u)$ we obtain

$$\begin{aligned} & (u + 1)f'_k + u(u + 2)f'_{k+1} + (\ell + 3 + k)f_k + (\ell + 4 + k)(u + 1)f_{k+1} \\ & = (-1)^{\ell+1} \frac{\delta^{(\ell+k+2)}}{(\ell + k + 2)!}, \quad k = 0, 1, \dots, \ell - 1. \end{aligned} \quad (54)$$

For finding a solution we introduce a suitable ansatz

$$f_k(u) = \sum_{p=0}^{k+1} g_k^p \frac{\delta^{(\ell+k+1-p)}(u)}{(\ell+k+1-p)!}. \quad (55)$$

Substituting (55) in (54) we obtain the following recursion equations for the unknown coefficients g_k^p

$$g_{k+1}^0 - g_k^0 = -\frac{1}{\ell+k+2}, \quad (56)$$

$$\begin{aligned} & \frac{g_{k+1}^{p+1}}{(\ell+k+1-2p)_{p+1}} - \frac{g_k^{p+1}}{(\ell+k-2p)_{p+1}} \\ &= -\frac{p+1}{(\ell+k-2p)_{p+2}} (g_{k+1}^p - g_k^p), \quad p = 1, 2, \dots, k, \end{aligned} \quad (57)$$

$$g_{k+1}^{k+1} - g_k^{k+1} = 0. \quad (58)$$

Now note that the following nontrivial expression solves the recursion (57) with initial condition (56) (see the proof in Appendix C)

$$\begin{aligned} g_k^p &= \frac{(-1)^p(p-1)!}{2^p(\ell+k+2-p)_p} \\ &+ \sum_{n=0}^{[p/2]} \frac{n!}{2^n} \binom{p}{n} \binom{p-n}{n} (\ell+k-2(p-n-1))_{p-2n} \Delta_{p-n}, \end{aligned} \quad (59)$$

where $\Delta_1, \dots, \Delta_\ell$ is a set of ℓ unknown constants. Substituting this solution in the so-called cutting conditions (58) we obtain ℓ linear equations for the unknown constants Δ_n in triangular form

$$\frac{(-1)^k}{(\ell+k+2)!} + \sum_{n=0}^{[k/2]} \frac{2^{k+1-n} \Delta_{k+1-n}}{n!} \binom{\ell}{k-2n} = 0. \quad (60)$$

So we prove that there is a consistent solution for the equations (54) and we manage that the singular part of the correlation function with an appropriate choice of the finite part f_k does not violate the gauge Ward identity. Of course this solution violates tracelessness of the loop function due to the existence of the finite part and we can say that we observed a higher spin version of the trace anomaly. The important point is the following: Even after the violation of tracelessness our theory is still in the framework of Fronsdal's [1] gauge invariance for massless but only double traceless gauge fields. In this formulation the conservation condition for the currents with the nonzero trace looks like

$$\nabla^\mu \frac{\partial}{\partial a^\mu} J^{(\ell)}(a; z) = O(a^2). \quad (61)$$

This corresponds to the usual conservation for the part of the currents expanded in I_1, I_2 bitensors which we actually checked here. Finally note that because the initial conditions (56) include only the difference between pairs of neighboring variables g_k^0 we have as one degree of freedom an arbitrary g_0^0 . This we can interpret as an arbitrary renormalization point.

5 Renormalization and RG equations

Now we return to (52)

$$F_k^R(u) = \binom{\ell}{k} \left(\frac{1}{\epsilon} \frac{(-1)^{\ell+1}}{(\ell+k+1)!} \delta^{(\ell+k+1)}(u) + f_k(u) \right) + F_k^{Ren}(u), \quad (62)$$

where the finite renormalization part is (55)

$$f_k(u) = \sum_{p=0}^{k+1} g_k^p \frac{\delta^{(\ell+k+1-p)}(u)}{(\ell+k+1-p)!}. \quad (63)$$

The set of unknown constants g_k^p we can find from the system of equations (56)-(58).

We note that because the initial conditions (56) include only the difference between pairs of neighboring variables g_k^0 we have as one degree of freedom an arbitrary $g_0^0 = (-1)^{\ell+1}\mu$. The parameter μ we can interpret as an arbitrary renormalization point. The important point here is the following: *Equation (56) leads to the special dependence of all the g_k^0 from μ*

$$g_k^0 = (-1)^{\ell+1}\mu + \tilde{g}_k^0, \quad \tilde{g}_0^0 = 0 \quad (64)$$

$$f_k(u) = (-1)^{\ell+1}\mu \frac{\delta^{(\ell+k+1)}(u)}{(\ell+k+1)!} + \sum_{p=1}^{k+1} \tilde{g}_k^p \frac{\delta^{(\ell+k+1-p)}(u)}{(\ell+k+1-p)!} \quad (65)$$

So we obtain the following dependence of the renormalized and singular parts on μ

$$F_k^R(u) = \binom{\ell}{k} \left(\left[\frac{1}{\epsilon} + \mu \right] \frac{(-1)^{\ell+1}}{(\ell+k+1)!} \delta^{(\ell+k+1)}(u) + \tilde{f}_k(u; \tilde{g}_k^p) \right) + F_k^{Ren}(u, \mu), \quad (66)$$

Then in a standard way we can derive the RG equations from the μ independence of the regularized set $\frac{d}{d\mu} F_k^R(u) = 0$

$$\frac{d}{d\mu} F_k^{Ren}(u, \mu) = \binom{\ell}{k} \frac{(-1)^\ell}{(\ell+k+1)!} \delta^{(\ell+k+1)}(u) \quad (67)$$

Using this and our integration procedure from the next section we can express the RG equations by the effective action

6 Singular parts of the effective action and integration

The singular part of the two point function of the higher spin currents in AdS_{d+1} space can be expressed in the following form

$$K(a, c; z_1, z_2) = \sum_{k=0}^{\ell} I_1^{\ell-k}(a, c) I_2^k(a, c) F_k^S(u; \mu) \quad (68)$$

$$F_k^S(u; \mu) = \binom{\ell}{k} \left(\left[\frac{1}{\epsilon} + \mu \right] \frac{(-1)^{\ell+1}}{(\ell+k+1)!} \delta^{(\ell+k+1)}(u) + \tilde{f}_k(u; \tilde{g}_k^p) \right) \quad (69)$$

$$\tilde{f}_k(u; \tilde{g}_k^p) = \sum_{p=1}^{k+1} \tilde{g}_k^p \frac{\delta^{(\ell+k+1-p)}(u)}{(\ell+k+1-p)!} \quad (70)$$

We are going to investigate the following integral

$$h^{(\ell)}(a; z_1) *_{a_1} K(a, c; z_1, z_2) *_{c_2} h^{(\ell)}(c; z_2) \quad (71)$$

$$*_{a_1} = \int \sqrt{g} d^4 z_1 \left[\overleftarrow{\partial}_a^\mu \overrightarrow{\partial}_\mu^a \right]^\ell = \int \sqrt{g} d^4 z_1 \hat{*}_a \quad (72)$$

First of all we have to express $\delta^{(n)}(u)$ as a combination of the covariant derivatives of the four dimensional delta function in the general coordinate system. It is possible because the parameter u is an AdS invariant variable. We start from the covariant delta function in the curved space with the metric $g_{\mu\nu}(z)$ and invariant measure $d\mu(z) = \sqrt{g} d^4 z$

$$\frac{\delta_{(4)}(z - a)}{\sqrt{g(z)}}, \quad \int \delta_{(4)}(z - a) d^4 z = 1 \quad (73)$$

In the polar coordinate system (see Appendix A) the invariant measure is

$$d\mu(z) = \sqrt{g} d^4 z = u(u+2) du d\Omega_3 \quad (74)$$

Therefore we can define

$$\frac{\delta_{(4)}(z - z_{pole})}{\sqrt{g(z)}} = \frac{\delta(u)}{u(u+2)\Omega_3} = -\frac{\delta^{(1)}(u)}{(u+2)\Omega_3} \quad (75)$$

$$u\delta^{(1)}(u) = -\delta(u) \quad (76)$$

Applying (A.33) we can derive

$$-(u+2)\square \frac{\delta^{(n)}(u)}{u+2} = 2n\delta^{(n+1)}(u) + [2 - n(n+1)]\delta^{(n)}(u) \quad (77)$$

which can be formulated as a recursion for the object $\phi_n(u) = \frac{\delta^{(n)}}{u+2}$

$$\phi_{n+1}(u) = -\hat{D}_n \phi_n(u) \quad (78)$$

$$\hat{D}_n = \frac{1}{2n} [\square + 2 - n(n+1)] \quad (79)$$

So it is easy to see that because $\phi_1(u) = -\frac{\delta_{(4)}(z-z_{pole})}{\sqrt{g(z)}} \Omega_3$ we can express the solution of (78) in the form

$$\phi_{n+1}(u) = (-1)^{n+1} \Omega_3 \left\{ \prod_{m=1}^n \hat{D}_m \right\} \frac{\delta_{(4)}(z-z_{pole})}{\sqrt{g(z)}} \quad (80)$$

Then using $\delta^{(n)}(u) = 2\phi_n - n\phi_{n-1}$ we obtain the final conversion formula

$$\delta^{(n)}(u) = (-1)^n \Omega_3 \left\{ 2\hat{D}_{n-1} + n \right\} \left\{ \prod_{m=1}^{n-2} \hat{D}_m \right\} \frac{\delta_{(4)}(z-z_{pole})}{\sqrt{g(z)}} \quad (81)$$

Now we concentrate on the singular and μ dependent part of (69)

$$[\frac{1}{\epsilon} + \mu] h^{(\ell)}(a; z_1) *_{a_1} \sum_{k=0}^{\ell} I_1^{\ell-k}(a, c) I_2^k(a, c) (-1)^{\ell+1} \binom{\ell}{k} \frac{\delta^{(\ell+k+1)}(u)}{(\ell+k+1)!} *_{c_2} h^{(\ell)}(c; z_2) \quad (82)$$

Admitting that our higher spin gauge field is transversal and traceless and integrating partially we obtain

$$[\frac{1}{\epsilon} + \mu] (-1)^{\ell+1} Z^\ell h^{(\ell)}(a; z_1) *_{a_1} I_1^\ell \delta^{(\ell+1)}(u) *_{c_2} h^{(\ell)}(c; z_2), \quad (83)$$

$$Z^\ell = \frac{1}{(\ell+1)} \sum_{k=0}^{\ell} \frac{(-1)^k}{(\ell-k)!(\ell+2)_k} = \frac{1}{(2\ell+1)\ell!} \quad (84)$$

Next we describe the way to take one integral in (83). Considering the following expression

$$\tilde{K}(a; z_1) = (-1)^{\ell+1} Z^\ell I_1^\ell \delta^{(\ell+1)}(u) *_{c_2} h^{(\ell)}(c; z_2), \quad (85)$$

we can using a conformal transformation fix the point z_1 as a pole for the coordinate system z_2 . In this case we can insert directly conversion formula (81) and obtain

$$\tilde{K}(a; z_{pole}) = Z^\ell \Omega_3 I_1^\ell \left\{ 2\hat{D}_\ell + (\ell+1) \right\} \left\{ \prod_{m=1}^{\ell-1} \hat{D}_m \right\} \frac{\delta_{(4)}(z_2 - z_{pole})}{\sqrt{g(z_2)}} *_{c_2} h^{(\ell)}(c; z_2) \quad (86)$$

Remembering the following formula (for transverse and traceless $h^{(\ell)}$)

$$\square I_1^\ell \hat{*}_{c_2} h^{(\ell)}(c; z_2) = I_1^\ell \hat{*}_{c_2} \{ \square + \ell \} h^{(\ell)}(c; z_2) \quad (87)$$

we obtain finally

$$\tilde{K}(a; z_{pole}) = Z^\ell \Omega_3 \left\{ (a^\mu c_\mu)^\ell \left\{ 2\hat{D}_\ell + (\ell + 2) \right\} \prod_{m=1}^{\ell-1} \left[\hat{D}_m + \frac{\ell}{2m} \right] \hat{*}_c h^{(\ell)}(c; z) \right\}_{z=z_{pole}} \quad (88)$$

With the help of this formula and (66) the singular part of the one loop effective action (82) for transversal and traceless external spin ℓ field can be expressed in the following local form

$$W_{Sing}^\ell(h^{(\ell)}, \mu) = \left[\frac{1}{\epsilon} + \mu \right] Z^\ell \Omega_3 \int \sqrt{g} d^4 z h_{\mu_1 \dots \mu_\ell}^{(\ell)} K^\ell(\square) h^{(\ell)\mu_1 \dots \mu_\ell} \quad (89)$$

$$K^\ell(\square) = \left\{ 2\hat{D}_\ell + (\ell + 2) \right\} \prod_{m=1}^{\ell-1} \left[\hat{D}_m + \frac{\ell}{2m} \right] \quad (90)$$

Then from (67) the scale anomaly (integrated trace anomaly) for the renormalized effective action comes out as

$$\begin{aligned} \frac{d}{d\mu} W_{Ren}^\ell(h^{(\ell)}, \mu) &= -\frac{d}{d\mu} W_{Sing}^\ell(h^{(\ell)}, \mu) \\ &= -Z^\ell \Omega_3 \int \sqrt{g} d^4 z h_{\mu_1 \dots \mu_\ell}^{(\ell)} K^\ell(\square) h^{(\ell)\mu_1 \dots \mu_\ell} \end{aligned} \quad (91)$$

In the case of $\ell = 2$ this integral should be proportional to the integrated square of the gravitational Weyl tensor linearized in the AdS_4 background (see [19] and ref. there)

$$C_{\lambda\rho}^{\mu\nu}(G) C_{\mu\mu}^{\lambda\rho}(G) = R_{\lambda\rho}^{\mu\nu}(G) R_{\mu\nu}^{\lambda\rho}(G) - 2R^{\mu\nu}(G) R_{\mu\nu}(G) + \frac{1}{3} R(G) R(G) \quad (92)$$

$$G_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}^{(2)} \quad , \quad \nabla^\mu h_{\mu\nu}^{(2)} = h_{\mu}^{(2)\mu} = 0 \quad (93)$$

For traceless and transversal $h_{\mu\nu}^{(2)}$ in an AdS_4 background (we put as before $L=1$) we have

$$R_{\lambda\rho}^{\mu\nu}(G) = R_{\lambda\rho}^{\mu\nu}(h^{(2)}) = 2\nabla^{[\mu} \nabla_{[\lambda} h_{\rho]}^{(2)\nu]} - 2\delta_{[\lambda}^{[\mu} h_{\rho]}^{(2)\nu]} \quad (94)$$

$$R_\lambda^\mu(h^{(2)}) = \frac{1}{2} \square h_\lambda^{(2)\mu} + h_\lambda^{(2)\mu} \quad , \quad R(h^{(2)}) = 0 \quad (95)$$

and straightforward calculations lead to

$$\int \sqrt{g} d^4 z C_{\lambda\rho}^{\mu\nu}(h^{(2)}) C_{\mu\mu}^{\lambda\rho}(h^{(2)}) = \frac{1}{2} \int \sqrt{g} d^4 z h_{\mu\nu}^{(2)} [\square^2 + 6\square + 8] h^{(2)\mu\nu}. \quad (96)$$

Then we can evaluate (90) for $\ell = 2$ and obtain

$$K^2(\square) = \frac{1}{4} [\square^2 + 6\square + 8]. \quad (97)$$

So we see that

$$W_{Sing}^2(h^{(\ell)}, \mu) = \left[\frac{1}{\epsilon} + \mu \right] \frac{Z^2 \Omega_3}{2} \int \sqrt{g} d^4 z C_{\lambda\rho}^{\mu\nu}(h^{(2)}) C_{\mu\nu}^{\lambda\rho}(h^{(2)}) \quad (98)$$

Finally note that if our external higher spin field is *on-shell* we can replace all Laplacians using the equation of motion

$$[\square + \ell] h^{(\ell)} = \Delta_\ell (\Delta_\ell - 3) h^{(\ell)}, \quad (99)$$

$$\square_a h^{(\ell)} = \nabla^\mu \frac{\partial}{\partial a^\mu} h^{(\ell)} = 0, \quad (100)$$

$$\Delta_\ell = \ell + 1. \quad (101)$$

Conclusions

In this article we considered the two point correlation function for traceless conserved higher spin currents in AdS_4 . Using a kind of dimensional regularization scheme we defined the correct renormalization procedure for the one loop diagram corresponding to this correlator and investigated the Ward identities. We have shown that extracting the delta function singularities we can define the renormalization in a gauge invariant way and obtain the violation of tracelessness. This means that we observed a *trace anomaly* for higher spin currents. This result was used for the derivation of the one loop anomalous effective action of the conformal scalar in AdS space that is minimally coupled to the higher spin external field or for the investigation of the one-loop renormalized propagators for the higher spin conformal gauge fields with linearized interaction with the conformal scalar.

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Appendix A

The Euclidian AdS_{d+1} metric

$$ds^2 = g_{\mu\nu}(z)dz^\mu dz^\nu = \frac{1}{(z^0)^2}\delta_{\mu\nu}dz^\mu dz^\nu \quad (\text{A.1})$$

can be realized as an induced metric for the hypersphere defined by the following embedding procedure in $d + 2$ dimensional Minkowski space

$$X^A X^B \eta_{AB} = -X_{-1}^2 + X_0^2 + \sum_{i=1}^d X_i^2 = -1, \quad (\text{A.2})$$

$$X_{-1}(z) = \frac{1}{2} \left(\frac{1}{z_0} + \frac{z_0^2 + \sum_{i=1}^d z_i^2}{z_0} \right), \quad (\text{A.3})$$

$$X_0(z) = \frac{1}{2} \left(\frac{1}{z_0} - \frac{z_0^2 + \sum_{i=1}^d z_i^2}{z_0} \right), \quad (\text{A.4})$$

$$X_i(z) = \frac{z_i}{z_0}. \quad (\text{A.5})$$

Using this embedding rules we can realize that the geodesic distance $\zeta(z, w)$ is just an $SO(1, d + 1)$ invariant scalar product

$$-X^A(z)Y^B(w)\eta_{AB} = \frac{1}{2z_0w_0} \left(2z_0w_0 + \sum_{\mu=0}^d (z - w)_\mu^2 \right) = \zeta, \quad (\text{A.6})$$

and therefore can be realized by a hyperbolic angle. Indeed we can introduce another embedding

$$X_{-1}(\Theta, \omega_\mu) = \cosh \Theta, \quad (\text{A.7})$$

$$X_\mu(\Theta, \omega_\mu) = \sinh \Theta \omega_\mu, \quad \sum_{\mu=0}^d \omega_\mu = 1, \quad (\text{A.8})$$

$$ds^2 = d\Theta^2 + \sinh^2 \Theta d\Omega_d. \quad (\text{A.9})$$

In these coordinates the geodesic distance between an arbitrary point $X^A(\Theta, \Omega_\mu)$ and the pole of the hypersphere $Y^A(\Theta = 0, \omega_\mu)$ is simply

$$\zeta = -X^A Y^B \eta_{AB} = \cosh \Theta. \quad (\text{A.10})$$

Therefore the invariant measure is expressed as

$$\sqrt{g} d\Theta d\Omega_d = (\sinh \Theta)^d d\Theta d\Omega_d = (\zeta^2 - 1)^{\frac{d-1}{2}} d\zeta d\Omega_d. \quad (\text{A.11})$$

So we see that the integration measure for $d = 3$ ($D = d + 1 = 4$) will cancel one order of $(\zeta - 1)^{-n}$ in short distance singularities and we have to count the singularities starting from $(\zeta - 1)^{-2}$.

In this article we use the following rules and relations for $\zeta(z, z')$, I_{1a} , I_{2c} and the bitensorial basis $\{I_i\}_{i=1}^4$

$$\square \zeta = (d + 1)\zeta, \quad \nabla_\mu \partial_\nu \zeta = g_{\mu\nu} \zeta, \quad g^{\mu\nu} \partial_\mu \zeta \partial_\nu \zeta = \zeta^2 - 1, \quad (\text{A.12})$$

$$\partial_\mu \partial_{\nu'} \zeta \nabla^\mu \zeta = \zeta \partial_{\nu'} \zeta, \quad \partial_\mu \partial_{\nu'} \zeta \nabla^\mu \partial_{\mu'} \zeta = g_{\mu'\nu'} + \partial_{\mu'} \zeta \partial_{\nu'} \zeta, \quad (\text{A.13})$$

$$\nabla_\mu \partial_\nu \partial_{\nu'} \zeta \nabla^\mu \zeta = \partial_\nu \zeta \partial_{\nu'} \zeta, \quad \nabla_\mu \partial_\nu \partial_{\nu'} \zeta = g_{\mu\nu} \partial_{\nu'} \zeta, \quad (\text{A.14})$$

$$\frac{\partial}{\partial a^\mu} I_{1a} \frac{\partial}{\partial a_\mu} I_{1a} = \zeta^2 - 1, \quad \frac{\partial}{\partial a^\mu} I_1 \frac{\partial}{\partial a_\mu} I_{1a} = \zeta I_{2c}, \quad (\text{A.15})$$

$$\frac{\partial}{\partial a^\mu} I_1 \frac{\partial}{\partial a_\mu} I_1 = c_2^2 + I_{2c}^2, \quad \frac{\partial}{\partial a^\mu} I_1 \frac{\partial}{\partial a_\mu} I_2 = \zeta I_{2c}^2, \quad \square_a I_4 = 2(d + 1)c_2^2, \quad (\text{A.16})$$

$$\frac{\partial}{\partial a^\mu} I_2 \frac{\partial}{\partial a_\mu} I_2 = (\zeta^2 - 1)I_{2c}^2, \quad \square_a I_3 = 2(d + 1)I_{2c}^2 + 2c_2^2(\zeta^2 - 1), \quad (\text{A.17})$$

$$\nabla^\mu \frac{\partial}{\partial a^\mu} I_1 = (d + 1)I_{2c}, \quad \nabla^\mu \frac{\partial}{\partial a^\mu} I_2 = (d + 2)\zeta I_{2c}, \quad \nabla^\mu I_1 \partial_\mu \zeta = I_2, \quad (\text{A.18})$$

$$\nabla^\mu \frac{\partial}{\partial a^\mu} I_3 = 4I_1 I_{2c} + 2(d + 2)\zeta c_2^2 I_{1a}, \quad \nabla^\mu I_2 \partial_\mu \zeta = 2\zeta I_2, \quad (\text{A.19})$$

$$\frac{\partial}{\partial a_\mu} I_1 \partial_\mu \zeta = \zeta I_{2c}, \quad \frac{\partial}{\partial a_\mu} I_2 \partial_\mu \zeta = (\zeta^2 - 1) I_{2c}, \quad \frac{\partial}{\partial a_\mu} I_1 \nabla_\mu I_1 = I_1 I_{2c}, \quad (\text{A.20})$$

$$\frac{\partial}{\partial a_\mu} I_1 \nabla_\mu I_2 = I_{2c} (\zeta I_1 + I_2) + c_2^2 I_{1a}, \quad \frac{\partial}{\partial a_\mu} I_2 \nabla_\mu I_1 = I_{2c} I_2, \quad (\text{A.21})$$

$$\frac{\partial}{\partial a_\mu} I_2 \nabla_\mu I_2 = 2\zeta I_{2c} I_2, \quad \nabla^\mu I_1 \nabla_\mu I_1 = a_1^2 I_{2c}, \quad \square I_1 = I_1, \quad (\text{A.22})$$

$$\nabla^\mu I_1 \nabla_\mu I_2 = I_2 I_1 + a_1^2 \zeta I_{2c}, \quad \square I_2 = (d+2)I_2 + 2\zeta I_1, \quad (\text{A.23})$$

$$\nabla^\mu I_2 \nabla_\mu I_2 = I_{2c}^2 + 2\zeta I_1 I_2 + a_1^2 I_{2c}^2 \zeta^2 + c_2^2 I_{1a}^2, \quad \nabla^\mu I_2 \partial_\mu \zeta = 2\zeta I_2, \quad (\text{A.24})$$

$$a^\mu \nabla_\mu I_{1a} = a^2 \zeta, \quad a^\mu \nabla_\mu I_{2c} = I_1, \quad a^\mu \nabla_\mu I_1 = a^2 I_{2c}, \quad (\text{A.25})$$

$$a^\mu \nabla_\mu I_2 = a^2 \zeta I_{2c} + I_{1a} I_1, \quad \nabla^\mu I_1 \partial_\mu \zeta = I_2. \quad (\text{A.26})$$

Using these relations we can derive ($F'_k := \frac{\partial}{\partial \zeta} F_k(\zeta)$)

- Divergence map

$$\nabla_1^\mu \frac{\partial}{\partial a^\mu} \Psi^\ell[F] = I_{2c} \Psi^{\ell-1}[Div_\ell F] + O(c_2^2), \quad (\text{A.27})$$

$$(Div_\ell F)_k = (\ell - k) \zeta F'_k + (k+1)(\zeta^2 - 1) F'_{k+1} \\ + (\ell - k)(\ell + d + k) F_k + (k+1)(\ell + d + k + 1) \zeta F_{k+1}. \quad (\text{A.28})$$

- Trace map

$$\square_a \Psi^\ell[F] = I_{2c}^2 \Psi^{\ell-2}[Tr_\ell F] + O(c_2^2), \quad (\text{A.29})$$

$$(Tr_\ell F)_k = (\ell - k)(\ell - k - 1) F_k + 2(k+1)(\ell - k - 1) \zeta F_{k+1} \\ + (k+2)(k+1)(\zeta^2 - 1) F_{k+2}. \quad (\text{A.30})$$

- Laplacian map

$$\square_1 \Psi^\ell[F] = \Psi^\ell[Lap_\ell F] + O(a_1^2, c_2^2), \quad (\text{A.31})$$

$$(Lap_\ell F)_k = (\zeta^2 - 1) F''_k + (d+1+4k) \zeta F'_k + [\ell + k(d+2\ell-k)] F_k \\ + 2\zeta(k+1)^2 F_{k+1} + 2(\ell - k + 1) F'_{k-1}, \quad (\text{A.32})$$

$$\square F_k(\zeta) = (\zeta^2 - 1) F''_k + (d+1) \zeta F'_k. \quad (\text{A.33})$$

- Gradient map

$$(a \cdot \nabla)_1 \Psi^\ell[F] = I_{1a} \Psi^\ell[Grad_\ell F] + O(a_1^2), \quad (\text{A.34})$$

$$(Grad_\ell F)_k = F'_k + (k+1) F_{k+1}. \quad (\text{A.35})$$

- Bigradient map

$$\Psi^{(\ell+1)}[G] = (a \cdot \nabla_1)(c \cdot \nabla_2) \Psi^{(\ell)}[F] + O(a_1^2, c_2^2), \quad (\text{A.36})$$

$$G_k = F''_{k-1}(\zeta) + (2k+1) F'_k(\zeta) + (k+1)^2 F_{k+1} \quad (\text{A.37})$$

Appendix B

Considering (35) as an ansatz we must solve (see (A.34)) the system of differential equations

$$F_{k,sing}^{(\ell)} \hat{=} G''_{k-1}(\zeta) + (2k+1)G'_k(\zeta) + (k+1)^2G_{k+1} \quad (\text{B.1})$$

($\hat{=}$ means: modulo regular terms). Since $G_k = 0$ for $k \geq \ell$, this system is solved recursively, starting with $k = \ell$, and lowering k step by step. The arbitrary polynomials of ζ obtained by integration are discarded since they are regular. The solution $\{G_k\}_{k=0}^{\ell-1}$ is therefore unique.

From (33) we obtain as solution

$$G_k \hat{=} \sum_{m=2}^{\ell+1} \sum_{n=1}^{\ell-k} \frac{a_{k+n,m}^{(\ell)}}{(m+k-1)_{n+1}} P_{n-1}(k)(\zeta-1)^{-m-k+1}, \quad (\text{B.2})$$

where $P_n(x)$ are polynomials of degree n defined by

$$P_n(x) = \frac{d}{dx}(x+1)_{n+1} = (x+1)_{n+1}[\psi(x+n+2) - \psi(x+1)]. \quad (\text{B.3})$$

By integration of a singular term $(\zeta-1)^{-2}$ we obtain a regular term $(\zeta-1)^{-1}$. This has happened in (B.2): The term $m=2$ is regular. Discarding this "leading regular term" and renumbering the sum we get

$$G_k \hat{=} \sum_{m=2}^{\ell} \sum_{n=1}^{\ell-k} \frac{a_{k+n,m+1}^{(\ell)}}{(m+k)_{n+1}} P_{n-1}(k)(\zeta-1)^{-m-k}, \quad (\text{B.4})$$

For $k=0$ we obtain a differential equation from (B.1) which acts as an integrability constraint

$$F_{0,sing}^{(\ell)} - G'_0 - G_1 \hat{=} 0. \quad (\text{B.5})$$

Inserting the expansion (35) and (B.2) in (B.4) we get from (B.5)

$$\sum_{k=0}^{\ell} \frac{k!}{(m)_k} a_{k,m}^{(\ell)} = 0 \quad \text{for all } 2 \leq m \leq \ell+1 \quad (\text{B.6})$$

Of course it is sufficient to prove

$$\sum_{k=0}^{\ell} \frac{k!}{(m)_k} Q_{k,r}^{(\ell)} = 0, \quad (\text{B.7})$$

for all $\{r, m | 0 \leq r \leq \ell+1-m \quad 2 \leq m \leq \ell+1\}$.

(B.7) can be verified easily for simple cases $m = \ell+1, r = 0$ or $m = \ell, r = 0$ or $r = 1$. In general we prove it by computer.

From (33) and (B.2) we derive an integration mapping for any fixed m which acts on $\left\{a_{k,m}^{(\ell)}\right\}_{k=0}^{\ell}$

$$a_{k,m}^{(\ell)} \rightarrow \tilde{b}_{k,m-1}^{(\ell-1)} = \sum_{n=1}^{\ell-k} \frac{a_{k+n,m}^{(\ell)}}{(m+k-1)_n} P_{n-1}(k), \quad (\text{B.8})$$

By trying on a computer one can show that this mapping can be repeated (with (ℓ, m) next replaced by $(\ell - 1, m - 1)$) exactly $n = m - 1$ times. The resulting coefficient is denoted

$$b_{k,n}^{(\ell-n)}, \quad (\text{B.9})$$

so that (36) holds. It depends in fact on all three parameters ℓ, k, n . For $n = \ell$ it can be proved that

$$b_{0,\ell}^{(0)} = 1, \quad (\text{B.10})$$

whereas for $n = \ell - 1$ we have guessed from a finite number of examples

$$b_{0,\ell-1}^{(1)} = -\frac{1}{2}(\ell-1)_4, \quad b_{1,\ell-1}^{(1)} = -(\ell)_2. \quad (\text{B.11})$$

A simple consequence of (36) and (B.10) is that the maximal singular terms in $\Psi^{(\ell)mixed}$ are

$$\Psi_{max.sing}^{(\ell)mixed} = [(a \cdot \nabla_1)(c \cdot \nabla_2)]^\ell (\zeta - 1)^{-1} \quad (\text{B.12})$$

Appendix C

We make an ansatz for g_k^p ($p \geq 1$)

$$g_k^p = \sum_{n=0}^{[p/2]} \alpha_{p,n} \Delta_{p-n} (\ell + k - 2(p - n - 1))_{p-2n} + \frac{(-1)^p \gamma_p}{(\ell + k + 2 - p)_p} \quad (\text{C.1})$$

and show that it is consistent with (57) if $\alpha_{p,n}, \gamma_p$ satisfy the recursion relations

$$\alpha_{p+1,n+1} = \frac{(p+1)(p-2n)}{2(n+1)} \alpha_{p,n}, \quad (\text{C.2})$$

$$\gamma_{p+1} = \frac{1}{2} p \gamma_p, \quad (\text{C.3})$$

which by the initial conditions

$$\begin{aligned} \alpha_{p,0} &= 1 && \text{(by definition)} \\ \gamma_1 &= \frac{1}{2} \end{aligned} \quad (\text{C.4})$$

imply

$$\alpha_{p,n} = \frac{1}{2^n} n! \binom{p}{n} \binom{p-n}{n}, \quad (\text{C.5})$$

$$\gamma_p = \frac{1}{2^p} (p-1)!. \quad (\text{C.6})$$

So we need to prove consistency of the ansatz (C.1) and the two recursion relations (C.2), (C.3).

Eqn. (57) is already written in a form that is suited for our strategy. Inserting (C.1) in the r.h.s. of (57), we obtain a difference equation for g_k^{p+1} of first order which is solved by summation. The result is compared with the ansatz. After insertion of (C.1) the r.h.s. of (57) is

$$-(p+1) \left\{ \sum_{n=0}^{[p/2]} \alpha_{p,n} (p-2n) \Delta_{p-n} [(\ell+k-2p)_{2n+3}]^{-1} - (-1)^p p \gamma_p [(\ell+k-2p)_{2p+3}]^{-1} \right\}. \quad (\text{C.7})$$

For the summation we replace k by k' and sum $\sum_{k'=0}^{k-1}$. In the first term of (C.7) we have

$$\begin{aligned} \sum_{k'=0}^{k-1} [(\ell+k'-2p)_{2n+3}] &= \frac{1}{2(n+1)} \left[\frac{(\ell-1-2p)!}{(\ell+1-2(p-n))!} \right. \\ &\quad \left. - \frac{(\ell+k-1-2p)!}{(\ell+k+1-2(p-n))!} \right]. \end{aligned} \quad (\text{C.8})$$

In the second term of (C.7) we have an analogous expression to sum with n replaced by p .

On the l.h.s. of (57) we have after summation

$$\frac{g_k^{p+1}}{(\ell+k-2p)_{p+1}} - \frac{g_0^{p+1}}{(\ell-2p)_{p+1}}. \quad (\text{C.9})$$

Since for $k=0$ there is no p allowed by (57), we cancel g_0^{p+1} against the k -independent terms (resulting from $k'=0$) on the r.h.s. Multiplying both sides with

$$(\ell+k-2p)_{p+1}, \quad (\text{C.10})$$

and using

$$\frac{(\ell+k-2p)_{p+1} (\ell+k-1-2p)!}{(\ell+k+1-2(p-n))!} = (\ell+k-2(p-n-1))_{p-2n-1}, \quad (\text{C.11})$$

the consistency of our ansatz and the correctness of (C.2), (C.3) are easily inspected.